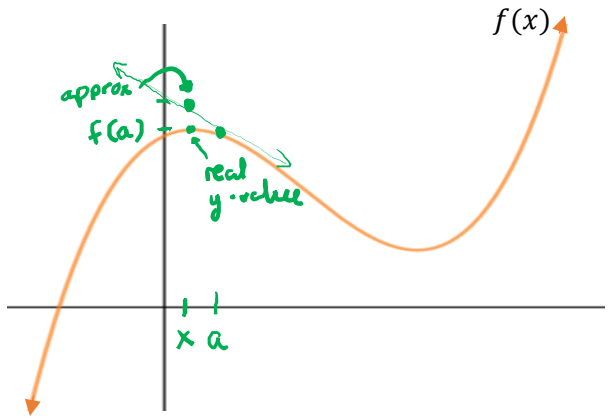
DATE: _____

Linearization (Linear Approximation)

<u>Linear Approximation</u>

→use of a tangent line at $(a, f(a))$ to approximate some y-value at some x-value.



If f is diff'able at $x = a$, then the equation of the tangent line at $(\underbrace{a}_{x_1}, \underbrace{f(a)}_{y_1})$ is:

$$y - y_1 = m(x - x_1)$$

$$f(x) - f(a) = f'(a)(x - a)$$

$$f(x) \approx f'(a)(x - a) + f(a)$$

or $L(x) \approx \underbrace{f'(a)(x-a) + f(a)}$

approximates $f(a)$ to be the $L(x)$ approximate y-value @ x.

the actual y-value @ x

Example 1:

Estimate $f(\overset{x}{4.1})$ for $f(x) = \sqrt{x^2 + 9}$.

$$y - y_1 = m(x - x_1)$$

$$f(x) - f(a) = f'(a)(x - a)$$

$$f(4.1) - f(4) = f'(4)(4.1 - 4)$$

$$f(4.1) \approx f'(4)(4.1 - 4) + f(4)$$

$$\approx \frac{4}{5}(.1) + 5$$

$$\approx .8(.1) + 5$$

$$\boxed{f(4.1) \approx 5.08}$$

Real y-value $f(4.1) = \sqrt{(4.1)^2 + 9}$
$= 5.080$ ☺

$a = 4$ #close to 4.1

$$f(4) = \sqrt{4^2 + 9}$$
$$= 5$$

$$f(x) = (x^2 + 9)^{1/2}$$

$$f'(x) = 2x \cdot \tfrac{1}{2}(x^2 + 9)^{-1/2}$$

$$f'(4) = 2(4) \cdot \tfrac{1}{2}(4^2 + 9)^{-1/2}$$

$$= 4 \cdot \frac{1}{\sqrt{25}}$$

$$= \frac{4}{5}$$

Example 2:

The function f is twice-differentiable with $\boxed{f(\overset{x_1}{2}) = 1}$, $\boxed{f'(2) = 4}$, and $f''(2) = 3$. What is the value of the approximation of $f(\overset{x}{1.9})$ using the line tangent to the graph of f at $x = 2$?

$y - y_1 = m(x - x_1)$

$f(1.9) - f(2) = f'(2)(1.9 - 2)$

$f(1.9) \approx f'(2)(1.9 - 2) + f(2)$

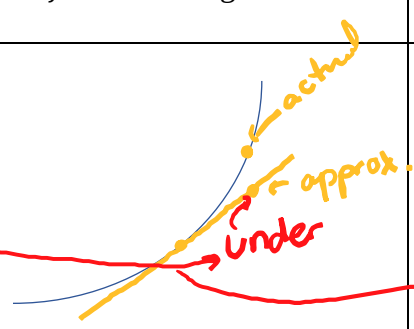
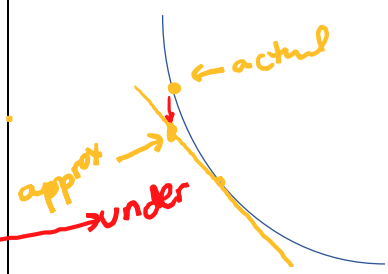
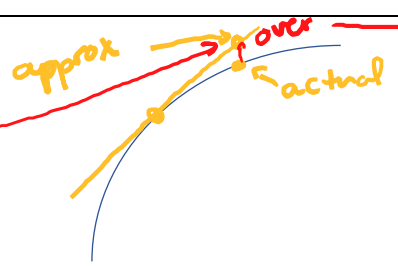
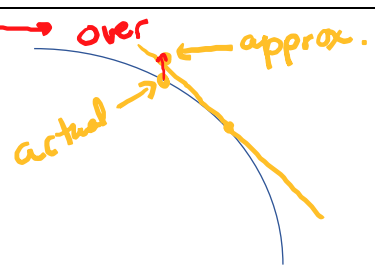
$\approx 4(-.1) + 1$

$\approx -.4 + 1$

$\boxed{f(1.9) \approx .6}$

Overapproximation vs. Underapproximation

Is the tangent line approximation an over or under approximation of the actual value?

	f is increasing	f is decreasing
f is concave up		
f is concave down		

Conclusion:

Tangent line approximation is an over approximation of the actual value when:

f is concave down (or $f'' < 0$ or f' dec)

Tangent line approximation is an under approximation of the actual value when:

f is concave up (or $f'' > 0$ or f' inc)

Example 1:

Is the tangent line approximation for $f(4.1)$ where $f(x) = \sqrt{x^2+9}$ an over or under approximation of the actual value of $f(4.1)$?

From above:

$$f'(x) = 2x \cdot \frac{1}{2}(x^2+9)^{-1/2}$$

$$= \frac{x}{(x^2+9)^{1/2}}$$

$$f''(x) = \frac{(x^2+9)^{1/2}(1) - x \cdot 2x \cdot \frac{1}{2}(x^2+9)^{-1/2}}{((x^2+9)^{1/2})^2}$$

$$= \frac{(x^2+9)^{1/2} - x^2(x^2+9)^{-1/2}}{x^2+9}$$

$$f''(4) = \frac{\sqrt{25} - 16 \cdot \frac{1}{\sqrt{25}}}{25}$$

$$= \frac{5 - 4/5}{25} > 0$$

$\therefore$ Tangent line approx of $f(4.1)$ is an underapprox of $f(4.1)$ b/c $f''(4) > 0$

Example 2:

The function f is twice-differentiable with $f(2) = 1$, $f'(2) = 4$, and $f''(2) = 3$. Is the approximation of $f(1.9)$ using the line tangent to the graph of f at $x = 2$ an overestimate or underestimate of $f(1.9)$?

$f''(2) = 3 > 0$

$\therefore$, approx of $f(1.9)$ is an underestimate of $f(1.9)$ b/c $f''(2) > 0$